Bell-CHSH Inequality and Genetic Algorithms

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We apply genetic algorithms to find the value where the CHSH inequality is violated.

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The Bell-CHSH (Clauser-Horne-Shimony-Holt) inequality [1 – 3] plays a central role in quantum mechanics for entangled states and local hidden variables theories. Here we show how genetic algorithms can be used to find values where the Bell-CHSH inequality is violated. In particular we want to find the values where the inequality is maximally violated.

Let $\boldsymbol{n}$, $\boldsymbol{m}$ be unit vectors in $\mathbf{R}^3$, i. e. $\|\boldsymbol{n}\| = \|\boldsymbol{m}\| = 1$. Let σ_1, σ_2, σ_3 be the Pauli spin matrices, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\boldsymbol{n} \cdot \boldsymbol{\sigma} = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$. Consider the spin singlet state (entangled state, Bell state)

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(\binom{1}{0} \otimes \binom{0}{1} - \binom{0}{1} \otimes \binom{1}{0}\right). \quad (1)$$

Calculating the quantum mechanical expectation values $E(\boldsymbol{n}, \boldsymbol{m})$

$$E(\boldsymbol{n}, \boldsymbol{m}) = \langle \Psi^- | (\boldsymbol{n} \cdot \boldsymbol{\sigma}) \otimes (\boldsymbol{m} \cdot \boldsymbol{\sigma}) | \Psi^- \rangle \quad (2)$$

yields

$$\begin{aligned} E(\boldsymbol{n}, \boldsymbol{m}) &= -\sum_{j=1}^{3} m_j n_j = -\boldsymbol{m} \cdot \boldsymbol{n} \\ &= -\|\boldsymbol{m}\| \cdot \|\boldsymbol{n}\| \cos\phi = -\cos\phi_{\boldsymbol{n},\boldsymbol{m}} , \end{aligned} \quad (3)$$

where ϕ is the angle ($\phi \in [0, \pi]$) between the two quantization directions $\boldsymbol{m}$ and $\boldsymbol{n}$. We write $\phi_{\boldsymbol{n},\boldsymbol{m}}$ to indicate that ϕ is the angle between $\boldsymbol{m}$ and $\boldsymbol{n}$. The Bell-CHSH inequality [1 – 3] is given by

$$|E(\boldsymbol{n}, \boldsymbol{m}) - E(\boldsymbol{n}, \boldsymbol{m}')| + |E(\boldsymbol{n}', \boldsymbol{m}') + E(\boldsymbol{n}', \boldsymbol{m})| \le 2. \quad (4)$$

Inserting (3) into (4) yields

$$|\cos\phi_{\boldsymbol{n},\boldsymbol{m}} - \cos\phi_{\boldsymbol{n},\boldsymbol{m}'}| + |\cos\phi_{\boldsymbol{n}',\boldsymbol{m}'} + \cos\phi_{\boldsymbol{n}',\boldsymbol{m}}| \le 2. \quad (5)$$

We want to find the angles, where the inequality is maximally violated. To apply genetic algorithms we use the form

$$\|\boldsymbol{n} \cdot \boldsymbol{m} - \boldsymbol{n} \cdot \boldsymbol{m}'\| + \|\boldsymbol{n}' \cdot \boldsymbol{m}' + \boldsymbol{n}' \cdot \boldsymbol{m}\| \le 2 \quad (6)$$

of the Bell-CHSH inequality. Genetic algorithms [4, 5] are the tool to be used to solve optimization problems in particular when the function to be optimized cannot be differentiated. We have to maximize the left-hand of equation (6), i. e. the function

$$f(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{n}', \boldsymbol{m}') = \|\boldsymbol{n} \cdot \boldsymbol{m} - \boldsymbol{n} \cdot \boldsymbol{m}'\| + \|\boldsymbol{n}' \cdot \boldsymbol{m}' + \boldsymbol{n}' \cdot \boldsymbol{m}\| \quad (7)$$

and find the values for unit vectors $\boldsymbol{m}$, $\boldsymbol{n}'$, $\boldsymbol{m}$, $\boldsymbol{m}'$. We express the unit vectors $\boldsymbol{n}$, $\boldsymbol{m}$, $\boldsymbol{n}'$, $\boldsymbol{m}'$ using spherical coordinates in $\mathbf{R}^3$, for example

$$\boldsymbol{n} = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2), \quad (8)$$

where $-\pi \le \theta_1 \le \pi$ and $0 \le \theta_2 \le \pi$. Hardy and Steeb [6] showed that genetic algorithms can be used directly with floating point numbers using mutation and crossing for the genetic operations. Applying these genetic operations provide us with a good approximation to the angles

$$\phi_{\boldsymbol{n},\boldsymbol{m}'} = 3\pi/4, \quad \phi_{\boldsymbol{n},\boldsymbol{m}} = \phi_{\boldsymbol{n}',\boldsymbol{m}'} = \phi_{\boldsymbol{n}',\boldsymbol{m}} = \pi/4, \quad (9)$$

where $\cos(3\pi/4) = -1/\sqrt{2}$ and $\cos(\pi/4) = 1/\sqrt{2}$. This leads to the maximal violation of the Bell-CHSH inequality which is given by $2\sqrt{2}$. Angles which violate the inequality (5) are called Bell angles.

An extension is the case where also the normalized states $|\phi\rangle$ have to be found. Since the states are normalized we use spherical coordinates in $\mathbf{R}^4$:

$$(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \\ \sin\theta_1\sin\theta_2\sin\theta_3)^{\mathrm{T}}$$

where $-\pi \leq \theta_1 \leq \pi$, $0 \leq \theta_j \leq \pi$ with $j = 2,3$. This includes all four Bell states of the form

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}.$$

Applying genetic algorithms provides a good approximation of the spin singlet state (1) given above. The extension to $\mathbf{C}^4$ would be straightforward by adding a phase to the components.

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